# Contest Problem Set 12306

## Team Round Problem 8

**David Sun**

Math League, LLC

## Problem

Let $S$ be the set of all positive integers $b$ for which $a^2 - b^2$ is equal to the square of an odd prime number for some positive integer $a$. Compute the greatest common divisor of all elements of $S$.

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$$b = \boxed{4} \cdot \frac{k \cdot (k+1)}{2}$$

# Concepts

- factor expressions
- reason about restrictions
- add and subtract equations
- for every two consecutive even integers, 4 divides one of them
- for every two consecutive integers, 2 divides one of them